Indian Statistical Institute
Semestral Examination
Differential Topology-BMath III

Max Marks: 60 Time: 3 hours

Throughout, X, Y, Z will denote manifolds without boundary unless otherwise stated. All maps are assumed to be smooth.

(1) (a) Let $p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$ be a non-constant polynomial with complex coeffients, and consider the associated map $z \mapsto p(z)$ of the complex plane $\mathbb{C} \longrightarrow \mathbb{C}$. Is it always true that p is a submersion except at finitely many points. Justify. [4]

(b) Show that $SL_n(\mathbb{R})$, the set of $n \times n$ matrices with determinant equal to 1, is a manifold. Determine its dimension. Describe the tangent space $T_A(SL_n(\mathbb{R}))$ at a matrix $A \in SL_n(\mathbb{R})$. [6]

(c) Define the notion of a manifold with boundary. Show that $X = [0,1] \times [0,1]$ is not a manifold with boundary. [5]

(2) (a) Let $f : X \longrightarrow Y$ be a map and Z a submanifold of Y. If $f \pitchfork Z$, show that $f^{-1}(Z)$ is a submanifold of X with $\mathrm{codim}(f^{-1}(Z)) = \mathrm{codim}(Z)$. [8]

(b) State the Stability theorem. Explain all the terms in the theorem. Let $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ be a map with $\rho(s) = 1$ if $|s| < 1$, $\rho(s) = 0$ if $|s| > 2$. Define $f_t : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_t(x) = x\rho(tx)$. Verify that the Stability theorem is false for noncompact domains by checking that the function ρ is a counterexample to all parts of the Stability theorem. [12]

(3) (a) Let $f : X \longrightarrow Y$ be a map and Z a submanifold of Y. Define the mod 2 intersection number $I_2(f, Z)$. Show that homotopic maps have the same mod 2 intersection number. [5]

(b) Suppose $f : X \longrightarrow Y$ is a diffeomorphism with X compact. Let Y_i, $i = 1, 2$ be submanifolds of Y such that $I_2(Y_1, Y_2)$ is defined. Let $X_i = f^{-1}(Y_i)$, $i = 1, 2$. Show that $I_2(X_1, X_2)$ is also defined and $I_2(X_1, X_2) = I_2(Y_1, Y_2)$. [10]

(c) State the Borsuk-Ulam theorem. Let $f_1, \dots, f_k : S^k \longrightarrow \mathbb{R}$ be antipode preserving maps. Show that there exists $x \in S^k$ such that $f_i(x) = 0$ for all $i = 1, 2, \dots, k$. [5]